arXiv:hep-th/0312045v1 4 Dec 2003

Phase spaces of Doubly Special Relativity

A. Błaut*, M. Daszkiewicz†, J. Kowalski–Glikman‡, and S. Nowak§

Institute for Theoretical Physics
University of Wrocław
Pl. Maxa Borna 9
Pl–50-204 Wrocław, Poland

February 7, 2008

Abstract

We show that depending on the direction of deformation of κ-Poincaré algebra (time-like, space-like, or light-like) the associated phase spaces of single particle in Doubly Special Relativity theories have the energy-momentum spaces of the form of de Sitter, anti-de Sitter, and flat space, respectively.

*email: ablaut@ift.uni.wroc.pl
†email: marcin@ift.uni.wroc.pl
‡email: jurekk@ift.uni.wroc.pl; Research partially supported by the KBN grant 5PO3B05620.
§email: pantera@ift.uni.wroc.pl

It has been claimed for some time that quantum groups could play significant role in quantizing gravity (see, for example, [1] and also [2], where the role of quantum symmetries in 2+1 quantum gravity has been explicitly shown.) If this claim is correct in the case of 3+1 quantum gravity, then one can expect that the symmetries of "quantum special relativity" defined as a flat space limit of quantum gravity should also be properly defined in terms of quantum groups.

Following this intuition quantum κ-Poincaré algebra [3], [4] has been proposed some time ago as a posible algebra of symmetries of flat space in Planckian regime. This algebra was incorporated later to the construction of Doubly Special Relativity (DSR) theories [5], [6], [7], [8], in which it plays a role of the symmetry algebra of one-particle states (another version of DSR theory has been proposed in [9], [10], but it can be also described in terms of a κ-Poincaré algebra [11].)

It happened to be more natural to introduce DSR as a theory of energy and momenta with deformed action of boosts. It turns out, however that one can make use of the co-product of κ-Poincaré algebra to construct position sector of DSR. In particular it has been shown in [12], [13], [14] that in DSR one particle dynamics takes place in the phase space, whose energy-momentum manifold forms four dimensional de Sitter space, while the space of positions is the non-commutative κ-Minkowski space [4]. The link between κ-Minkowski space and κ-Poincaré algebra, which was originally obtained by using duality of Hopf algebra ([4], [15]), has been recently re-derived in the paper [16] starting from dynamics on κ-Minkowski space.

In the analysis reported in these papers the starting point was a particular form of κ-Poincaré algebra, with deformation along time-like direction. It is well known however that there are other κ-Poincaré algebras [17], [18] in which the deformation can be directed along light-like or space-like directions. In this paper we will present construction of phase spaces in such a general case.

Let us start with reviewing the general form of κ-Poincaré algebra (in the case of arbitrary Minkowski metric $g^{\mu\nu}$), whose algebraic part is given by [17]

$$[M^{\mu\nu}, M^{\alpha\beta}] = i\left(g^{\mu\beta}M^{\nu\alpha} - g^{\nu\beta}M^{\mu\alpha} + g^{\nu\alpha}M^{\mu\beta} - g^{\mu\alpha}M^{\nu\beta}\right) \tag{1}$$

$$[P_\mu, P_\nu] = 0, \quad [M^{ij}, P_0] = 0 \tag{2}$$

$$
\begin{aligned}
[M^{ij}, P_k] &= i\kappa \left(\delta^j_k g^{0i} - \delta^i_k g^{0j}\right)\left(1 - \mathrm{e}^{-P_0/\kappa}\right) + i\left(\delta^j_k g^{is} - \delta^i_k g^{js}\right) P_s \quad (3)\\
[M^{i0}, P_0] &= i\kappa g^{i0}\left(1 - \mathrm{e}^{-P_0/\kappa}\right) + i g^{ik} P_k \quad (4)\\
[M^{i0}, P_k] &= -i\frac{\kappa}{2} g^{00}\delta^i_k \left(1 - \mathrm{e}^{-2P_0/\kappa}\right) - i\delta^i_k g^{0s} P_s \mathrm{e}^{-P_0/\kappa}\\
&\quad + i g^{0i} P_k \left(\mathrm{e}^{-P_0/\kappa} - 1\right) + \frac{i}{2\kappa}\delta^i_k g^{rs} P_r P_s - \frac{i}{\kappa} g^{is} P_s P_k \quad (5)
\end{aligned}
$$

Here Greek indices run from 0 to D, where $D+1$ is the dimension of spacetime under consideration, while Latin from 1 to D. It is easy to see that the standard κ-Poincaré algebra of DSR in the bicrossproduct basis is a particular example of the general algebra (1)–(5), corresponding to the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ (in four spacetime dimensions) so that the deformation is in the time-like direction. One can consider other cases, however, with $g_{\mu\nu} = \mathrm{diag}(1, -1, 1, 1)$, and

$$
g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)
$$

corresponding to space-like and light-like [17], [18] deformation, respectively. It should be stressed that analogous construction can be made in any dimension $D \geq 1$.

According to [17] the algebra (1)–(5) with the following co-product structure

$$
\begin{aligned}
\Delta(P_0) &= 1 \otimes P_0 + P_0 \otimes 1 \quad (7)\\
\Delta(P_k) &= P_k \otimes \mathrm{e}^{-P_0/\kappa} + 1 \otimes P_k \quad (8)\\
\Delta(M^{ij}) &= M^{ij} \otimes 1 + 1 \otimes M^{ij} \quad (9)\\
\Delta(M^{i0}) &= 1 \otimes M^{i0} + M^{i0} \otimes \mathrm{e}^{-P_0/\kappa} - \frac{1}{\kappa} M^{ij} \otimes P_j \quad (10)
\end{aligned}
$$

and appropriately given antipode defines Hopf algebra, which we call quantum κ-Poincaré algebra with arbitrary metric $g_{\mu\nu}$. To extend this structure to the whole of the phase space of a one-particle system one adds the dual quantum group with generators being Lorentz transformations $\Lambda^\mu{}_\nu$ and translations of momenta, which can be interpreted as positions X^μ. Their co-products are [17]

$$
\Delta(X^\mu) = \Lambda^\mu{}_\nu \otimes X^\nu + X^\mu \otimes 1
$$

and

$$\Delta(\Lambda^{\mu}{}_{\nu}) = \Lambda^{\mu}{}_{\rho} \otimes \Lambda^{\rho}{}_{\nu}$$

Next one defines the pairings between elements of the algebra and the group as follows

$$\begin{aligned}
\langle P_{\mu}, X^{\nu} \rangle &= i\delta^{\nu}_{\mu} \\
\left\langle \Lambda^{\mu}{}_{\nu}, M^{\alpha\beta} \right\rangle &= i\left(g^{\alpha\mu}\delta^{\beta}_{\nu} - g^{\beta\mu}\delta^{\alpha}_{\nu}\right) \\
\langle \Lambda^{\mu}{}_{\nu}, 1 \rangle &= \delta^{\mu}_{\nu}
\end{aligned}$$

The phase space brackets can be found by employing Heisenberg double procedure [19], [20], [12]

$$\begin{aligned}
[X^{\mu}, P_{\nu}] &= P_{\nu(1)} \left\langle X^{\mu}_{(1)}, P_{\nu(2)} \right\rangle X^{\mu}_{(2)} - P_{\nu} X^{\mu}, \\
[X^{\mu}, M^{\rho}{}_{\sigma}] &= M^{\rho}_{(1)\sigma} \left\langle X^{\mu}_{(1)}, M^{\rho}_{(2)\sigma} \right\rangle X^{\mu}_{(2)} - M^{\rho}{}_{\sigma} X^{\mu},
\end{aligned}$$

where we make use of the standard ("Sweedler") notation for co-product

$$\Delta\mathcal{T} = \sum \mathcal{T}_{(1)} \otimes \mathcal{T}_{(2)}.$$

In this way one gets the following phase space commutator constituting the so called κ-Minkowski, non-commutative spacetime

$$[X^{\mu}, X^{\nu}] = -\frac{i}{\kappa} X^{\mu}\delta^{\nu}_{0} + \frac{i}{\kappa} X^{\nu}\delta^{\mu}_{0} \tag{11}$$

$$[X^{\mu}, M^{k0}] = i\left(g^{k\mu}X^{0} - g^{0\mu}X^{k}\right) - \frac{i}{\kappa}\left(\delta^{\mu}_{0}M^{k0} + \delta^{\mu}_{l}M^{kl}\right) \tag{12}$$

$$[X^{\mu}, M^{kl}] = i\left(g^{k\mu}X^{l} - g^{l\mu}X^{k}\right) \tag{13}$$

and the commutators between positions X and momenta

$$[P_{\nu}, X^{\mu}] = -i\delta^{\mu}_{\nu} + \frac{i}{\kappa}\delta^{\mu}_{0} P_{\sigma}\left(\delta^{\sigma}_{\nu} - \delta^{0}_{\nu}\delta^{\sigma}_{\nu}\right). \tag{14}$$

It is interesting to ask which algebra the generators X^{μ} and $M^{\mu\nu}$ form. Because of (2)–(5) and (14) this $(D+1)(D+2)/2$-dimensional algebra is the algebra of symmetries of the $D+1$ dimensional energy-momentum space. Assuming that the symmetries act on the energy-momentum space transitively, and that the subalgebra of symmetries leaving invariant the point $P_{\mu} = 0$ is the Lorentz subalgebra, we see that energy-momentum space is isomorphic

to the quotient of the group generated by X and M algebra by its Lorentz subgroup. Therefore knowing the algebra of X and M we know the form of energy-momentum manifold.

To find the algebra in question let us make the identification

$$\hat{M}^{\mu\nu} = M^{\mu\nu}, \quad \hat{M}^{(D+1)\mu} = -\hat{M}^{\mu(D+1)} = \kappa X^{\mu}$$

One can easily check that the algebra of $(D+1)(D+2)/2$ generators $\hat{M}^{AB}$, $(A, B = 0, \ldots D+1)$ satisfying

$$[\hat{M}^{AB}, \hat{M}^{CD}] = i\left(\hat{g}^{BC}\hat{M}^{AD} - \hat{g}^{AC}\hat{M}^{BD} - \hat{g}^{BD}\hat{M}^{AC} + \hat{g}^{AD}\hat{M}^{BC}\right) \tag{15}$$

is equal to (11)–(13) if

$$\hat{g}^{AB} = \begin{pmatrix} & & & 1 \\ & & & 0 \\ & g^{\mu\nu} & & \vdots \\ & & & 0 \\ 1 & 0\cdots 0 & & 0 \end{pmatrix}$$

where

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & \cdots & g^{0D} \\ \vdots & \ddots & \vdots \\ g^{D0} & \cdots & g^{DD} \end{pmatrix}$$

is the Minkowski spacetime metric in the Hopf algebra (1)–(10).

First we ask if the metric $\hat{g}^{AB}$ can be degenerate, i.e., if it can have zero eigenvalues. Since the metric $g^{\mu\nu}$ is non-degenerate (with eigenvalues $-1, 1, \ldots, 1$) from standard theorems of matrix algebra it follows that the rank of $\hat{g}^{AB}$ can be at least $D+1$, and thus it can have at most one zero eigenvalue. This happens if, for example, one of the first $D+1$ columns of $\hat{g}^{AB}$ is proportional to its last column, i.e., in the light-like case. Straightforward algebraic considerations show next that $\hat{g}^{AB}$ can have only the following signatures: $[-, +, \ldots, +]$, $[-, -, +, \ldots, +]$ or $[0, -, +, \ldots, +]$ corresponding to the time-like, space-like and light-like deformations, respectively.

Let us now make this explicit in the four-dimensional case.

Time-like deformation. In this case $g^{\mu\nu} = \text{diag}\,(-1, 1, 1, 1)$ and the algebra (15) is the $SO(1,4)$ algebra, while the energy-momentum space is de

Sitter space $SO(1,4)/SO(1,3)$. This case has been analyzed before in [13] and [14].

Space-like deformation. $g^{\mu\nu} = \text{diag}\,(1,-1,1,1)$ and the algebra (15) is the $SO(2,3)$ algebra. The energy-momentum space is anti-de Sitter space $SO(2,3)/SO(1,3)$. It is worth noticing that anti-de Sitter space is the energy-momentum manifold of 2+1 dimensional quantum gravity coupled to point particle [22]. Explicitly, the relation between the generators J^μ (rotation and boosts) and y^μ (translations) employed in that paper, which satisfy the algebra

$$
\begin{aligned}
[y^\mu, y^\nu] &= 2\epsilon^{\mu\nu}{}_\sigma y^\sigma \\
[J^\mu, J^\nu] &= -\epsilon^{\mu\nu}{}_\sigma J^\sigma \\
[J^\mu, y^\nu] &= -\epsilon^{\mu\nu}{}_\sigma y^\sigma
\end{aligned}
$$

and our generators is the following

$$
\begin{aligned}
J^1 &= -i\kappa X^0 \\
J^2 &= iM^{01} \\
J^0 &= i\kappa X^1 - iM^{01} \\
y^1 &= i\kappa X^0 + iM^{12} \\
y^2 &= -i\kappa X^2 - iM^{01} + iM^{02} \\
y^0 &= -i\kappa X^1 + iM^{01} - M^{02}
\end{aligned}
$$

The relation of 2+1 quantum gravity to DSR theories and its possible relevance for 3+1 dimensional physics has been analyzed in [23] and [24].

Light-like deformation. Now the metric $g^{\mu\nu}$ is given by (6) and it is clear that the metric $\hat{g}^{AB}$ is degenerate. In context of DSR theory such deformation has been analyzed in [21]. It is easy to see that in this case the algebra (15) is the standard Poincaré algebra and that the energy-momentum space is the flat Minkowski space. Explicitly, the four commuting elements $\mathcal{X}^\mu$ have the form

$$
\mathcal{X}^0 = X^0 - \frac{1}{\kappa}\, M^{10}, \quad \mathcal{X}^1 = X^1, \quad \mathcal{X}^\alpha = X^\alpha - \frac{1}{\kappa}\, M^{1\alpha}, \;\; \alpha = 2,3.
$$

This case certainly deserves further studies, as it may correspond to DSR theories with commuting spacetime. There is a hope therefore that, for example, the construction of κ-deformed field theory [18] might be much simpler here

than in other cases. We will present the results of this investigations in a separate paper.

In conclusions let us recapitulate the results of reported above. We found that in general case, there are three DSR phase spaces associated with κ-Poincaré algebra. Let us stress that since this algebra has the same form in any dimension, our results hold for any space dimension $D \geq 1$ as well. It is of course an open problem which of this cases (if any) is a correct setting for 3+1 quantum special relativity. Work in this direction is in progress.

Acknowledgement. We would like to thank P. Ługiewicz and J. Lukierski for discussion and useful comments.

References

[1] S. Major and L. Smolin, "Quantum deformation of quantum gravity," Nucl. Phys. B **473** (1996) 267 [arXiv:gr-qc/9512020].

[2] F. A. Bais, N. M. Muller and B. J. Schroers, "Quantum group symmetry and particle scattering in (2+1)-dimensional quantum gravity," Nucl. Phys. B **640** (2002) 3 [arXiv:hep-th/0205021].

[3] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, "Q deformation of Poincare algebra," Phys. Lett. B **264** (1991) 331.

[4] S. Majid and H. Ruegg, "Bicrossproduct structure of kappa Poincare group and noncommutative geometry," Phys. Lett. B **334** (1994) 348 [arXiv:hep-th/9405107]; J. Lukierski, H. Ruegg and W. J. Zakrzewski, "Classical and quantum mechanics of free kappa relativistic systems," Annals Phys. **243** (1995) 90 [arXiv:hep-th/9312153].

[5] G. Amelino-Camelia, "Testable scenario for relativity with minimum-length," Phys. Lett. B **510** (2001) 255 [arXiv:hep-th/0012238].

[6] G. Amelino-Camelia, "Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale," Int. J. Mod. Phys. D **11** (2002) 35 [arXiv:gr-qc/0012051].

[7] J. Kowalski-Glikman, "Observer independent quantum of mass," Phys. Lett. A **286** (2001) 391 [arXiv:hep-th/0102098].

[8] N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, "Deformed boost transformations that saturate at the Planck scale," Phys. Lett. B **522** (2001) 133 [arXiv:hep-th/0107039].

[9] J. Magueijo and L. Smolin, "Lorentz invariance with an invariant energy scale," Phys. Rev. Lett. **88** (2002) 190403 [arXiv:hep-th/0112090].

[10] J. Magueijo and L. Smolin, "Generalized Lorentz invariance with an invariant energy scale," Phys. Rev. D **67** (2003) 044017 [arXiv:gr-qc/0207085].

[11] J. Kowalski-Glikman and S. Nowak, "Doubly special relativity theories as different bases of kappa-Poincare algebra," Phys. Lett. B **539** (2002) 126 [arXiv:hep-th/0203040].

[12] J. Kowalski-Glikman and S. Nowak, "Non-commutative space-time of doubly special relativity theories," Int. J. Mod. Phys. D **12** (2003) 299 [arXiv:hep-th/0204245].

[13] J. Kowalski-Glikman, "De Sitter space as an arena for doubly special relativity," Phys. Lett. B **547** (2002) 291 [arXiv:hep-th/0207279].

[14] J. Kowalski-Glikman and S. Nowak, "Doubly special relativity and de Sitter space," Class. Quant. Grav. **20** (2003) 4799 [arXiv:hep-th/0304101].

[15] S. Zakrzewski, J. Phys. **A 27**, 2075 (1994).

[16] A. Agostini, G. Amelino-Camelia and F. D'Andrea, "Hopf-algebra description of noncommutative-spacetime symmetries," arXiv:hep-th/0306013.

[17] P. Kosinski and P. Maslanka, "The κ-Weyl group and its algebra", arXiv: q-alg/9512018; in *From Field Theory to Quantum Groups*, ed. B. Jancewicz, J. Sobczyk, p. 41, World Scientific (1996).

[18] P. Kosinski, P. Maslanka, J. Lukierski and A. Sitarz, "Generalized kappa-deformations and deformed relativistic scalar fields on noncommutative Minkowski space," arXiv:hep-th/0307038.

[19] A. Nowicki, "Kappa-Deformed Phase Space and Uncertainty Relations," arXiv:math.QA/9803064.

[20] J. Lukierski and A. Nowicki, Proceedings of Quantum Group Symposium at Group 21, (July 1996, Goslar) Eds. H.-D. Doebner and V.K. Dobrev, Heron Press, Sofia, 1997, p. 186.

[21] A. Błaut, M. Daszkiewicz and J. Kowalski-Glikman, "Doubly special relativity with light-cone deformation," Mod. Phys. Lett. A **18** (2003) 1711 [arXiv:hep-th/0302157].

[22] H. J. Matschull and M. Welling, "Quantum mechanics of a point particle in 2+1 dimensional gravity," Class. Quant. Grav. **15** (1998) 2981 [arXiv:gr-qc/9708054].

[23] G. Amelino-Camelia, L. Smolin and A. Starodubtsev, "Quantum symmetry, the cosmological constant and Planck scale phenomenology," arXiv:hep-th/0306134.

[24] L. Freidel, J. Kowalski-Glikman and L. Smolin, "2+1 gravity and doubly special relativity," arXiv:hep-th/0307085.